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UNSYMMETRICAL BUCKLING
OF THIN SHALLOW SPHERICAL SHELLS

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INTRODUCTION

This paper is concerned with the theoretical study of buckling of clamped shallow spherical shells under uniform external pressure (Figure 1). For sufficiently large deflection, deformations of such shells are not proportional to the applied pressure. The shell deforms axisymmetrically under sufficiently low pressure and if we assume that the deflection remains axisymmetrical, the pressure-deflection relation may be represented by a curve such as OAB in Figure 2. At the maximum pressure q_{cr} , the shell tends to jump from A to B . So q_{cr} is the buckling pressure for axisymmetrical snapping. The problem of axisymmetrical snapping has been solved by different numerical methods in References (1-4) and the results agree with each other and are represented by the curve in Figure 3. As shown in Figure 3, the buckling pressures obtained in such a manner are too high as compared with experimental results obtained in References (5-6). Initial imperfections of the shell and unsymmetrical buckling are presumed to be the sources of this discrepancy between axisymmetrical buckling theory and experiment. Unsymmetrical deformation could conceivably start to develop at point C in Figure 2. After unsymmetrical deformation has been superimposed, the pressure-deflection curve might be represented by a branch CD or CE at C as shown. The pressure at C is defined as the critical

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pressure for unsymmetrical buckling. Grigolyuk[7]* applied the Galerkin method
to solve the problem of unsymmetrical buckling, but no numerical results were
given in his work. Gjelsvik and Bodner[8] and Parmerter and Fung[9] worked on
this problem by the energy method and numerical calculations were carried out
only for a particular unsymmetrical buckling mode. Weinitschke[10] used a power
series method to solve this problem and obtained extensive numerical results,
but, as will be shown, the buckling pressures obtained in his work are in serious
disagreement with the results of this paper.

BASIC EQUATIONS OF SHALLOW SPHERICAL SHELLS

A shell is called "thin" if the ratio of its thickness to the radius of
curvature of its middle surface is much less than unity; and a spherical shell
is called "shallow" if the ratio of its rise at the center to the base diameter
is less than, say, 1/8 . The middle surface of a shallow spherical shell can
be represented by the paraboloid

$$z = H[1-(\frac{r}{a})^2] \qquad (1)$$

where H is the rise of the middle surface at the center and a is the base
radius, as shown in Figure 1. The radius of curvature of the shell is

$$R = \frac{a^2}{2H} \qquad (2)$$

The stress resultants and bending moments per unit length of the shell are
shown in Figure 4, where N_r , N_θ and $N_{r\theta}$ are membrane forces respectively;

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* Numbers in the raised parenthesis refer to the references at the end of
this report.

Q_r and Q_θ are transverse shears and M_r , M_θ and $M_{r\theta}$ are meridional, circumferential and twisting moments respectively. Let U , V and W be the horizontal radial, horizontal tangential and vertical components of displacement respectively (Fig. 4) and let q be the external pressure which can be considered to be vertical.

In the following equations, we use the notations

$$(\ )' = \frac{\partial}{\partial r}(\ ) \quad \text{and} \quad (\dot{\ }) = \frac{\partial}{\partial \theta}(\ )$$

Equilibrium of moments requires

$$(rM_r)' + \dot{M}_{r\theta} - M_\theta - rQ_r = 0 \tag{3}$$

$$(rM_{r\theta})' + \dot{M}_\theta + M_{r\theta} - rQ_\theta = 0 \tag{4}$$

Equilibrium of stress resultants requires

$$(rN_r)' + \dot{N}_{r\theta} - N_\theta = 0 \tag{5}$$

$$(rN_{r\theta})' + \dot{N}_\theta + N_{r\theta} = 0 \tag{6}$$

$$[rN_r(W-z)' + N_{r\theta}\overline{(\dot{W-z})} + (rQ_r)']' + [\frac{1}{r} N_\theta \overline{(\dot{W-z})} + N_{r\theta}(W-z)' + Q_\theta] + rq = 0 \tag{7}$$

Note that nonlinearity has been introduced in Eq. (7) by considering the influence of W .

The stress strain relations are

$$\epsilon_r = \frac{1}{tE}(N_r - \nu N_\theta) \tag{8}$$

$$\epsilon_\theta = \frac{1}{tE}(N_\theta - \nu N_r) \tag{9}$$

$$\gamma_{r\theta} = \frac{2(1+\nu)}{tE} N_{r\theta} \tag{10}$$

$$M_r = D(\varkappa_r + \nu \varkappa_\theta) \tag{11}$$

$$M_\theta = D(\varkappa_\theta + \nu \varkappa_r) \tag{12}$$

$$M_{r\theta} = (1-\nu)D \varkappa_{r\theta} \tag{13}$$

where E and ν are the elastic modulus and Poisson's ratio respectively; t stands for the thickness of the shell and $D = \dfrac{Et^3}{12(1-\nu^2)}$.

The strain displacement relations are also nonlinear

$$e_r = U' - z'W' + \frac{1}{2}(W')^2 \tag{14}$$

$$e_\theta = \frac{1}{r} U + \frac{1}{r} \dot{V} - \frac{1}{r^2} \dot{z}\dot{W} + \frac{1}{2}(\frac{1}{r} \dot{W})^2 \tag{15}$$

$$\gamma_{r\theta} = \frac{1}{r} \dot{U} - \frac{1}{r} V + V' - \frac{1}{r} \dot{z}W' - \frac{1}{r} z'\dot{W} + \frac{1}{r} \dot{W}W' \tag{16}$$

$$\varkappa_r = -W'' \tag{17}$$

$$\varkappa_\theta = - \frac{1}{r^2} \ddot{W} - \frac{1}{r} W' \tag{18}$$

$$\varkappa_{r\theta} = -(\frac{1}{r} \dot{W})' \tag{19}$$

Equations (5) and (6) can be satisfied by setting

$$N_r = \frac{1}{r} F' + \frac{1}{r^2} \ddot{F} \tag{20}$$

$$N_\theta = F'' \tag{21}$$

$$N_{r\theta} = -(\frac{1}{r} \dot{F})' \tag{22}$$

where F is a stress function.

Eliminating the transverse shears Q_r and Q_θ from Eqs. (3), (4) and (7), we have

$$[(rM_r)' + \dot{M}_{r\theta} - M_\theta]' + \frac{1}{r} \overline{[(rM_{r\theta})' + \dot{M}_\theta + M_{r\theta}]} + rN_r(W-z)'' + N_\theta[\frac{1}{r}\overline{(W-z)} + (W-z)']$$

$$+ 2N_{r\theta}[\overline{(W-z)}' - \frac{1}{r}\overline{(W-z)}] + q_r = 0 \tag{23}$$

Then the equation of equilibrium obtained by direct substitution is

$$D\nabla^2\nabla^2 W = (\frac{1}{r} F' + \frac{1}{r^2} \ddot{F})(W-z)'' + \frac{1}{r^2} F''(\overline{W-z}) + \frac{1}{r} F''(W-z)'$$

$$+ 2(\frac{1}{r} \dot{F}' - \frac{1}{r^2} \dot{F})(\frac{1}{r} \dot{z}' - \frac{1}{r} \dot{W}' - \frac{1}{r^2} \dot{z} + \frac{1}{r^2} \dot{W}) + q \quad (24)$$

where $\nabla^2( ) \equiv ( )'' + \frac{1}{r}( )' + \frac{1}{r^2}(\ddot{\ })$.

Using Eqs. (8)-(10) and (20)-(22), we can show that

$$\frac{1}{r}(r\epsilon_\theta)'' - \frac{1}{r} \epsilon_r' - \frac{1}{r^2}(r\dot{\gamma}_{r\theta})' + \frac{1}{r^2} \ddot{\epsilon}_r = \frac{1}{Et} \nabla^2(N_r + N_\theta) = \frac{1}{Et} \nabla^4 F \quad (25)$$

Substituting Eqs. (14)-(16) into the left hand side, we get another fundamental equation involving F and ω

$$\nabla^4 F = Et \left\{ -2(\frac{1}{r} \dot{z}' - \frac{1}{r^2} \dot{z})(\frac{1}{r} \dot{W}' - \frac{1}{r^2} \dot{W}) + z''(\frac{1}{r} W' + \frac{1}{r^2} \ddot{W}) + W''(\frac{1}{r} z' + \frac{1}{r^2} \ddot{z}) \right.$$

$$\left. + (\frac{1}{r} \dot{W}' - \frac{1}{r^2} \dot{W})^2 - W''(\frac{1}{r} W' + \frac{1}{r^2} \ddot{W}) \right\} \quad (26)$$

which is the compatibility equation of shallow spherical shells.

Substituting Eq. (1) into Eqs. (24) and (26), we get Marguere's nonlinear differential equations[11] in polar coordinates for shallow spherical shells under uniform pressure,

$$D\nabla^4 W = \frac{1}{R} \nabla^2 F + (\frac{1}{r} F' + \frac{1}{r^2} \ddot{F})W'' + (\frac{1}{r} W' + \frac{1}{r^2} \ddot{W})F'' - 2(\frac{1}{r} \dot{F}' - \frac{1}{r^2} \dot{F})(\frac{1}{r} \dot{W}' - \frac{1}{r^2} \dot{W}) + q$$

$$(27)$$

$$\nabla^4 F = Et[-\frac{1}{R} \nabla^2 W + (\frac{1}{r} \dot{W}' - \frac{1}{r^2} \dot{W})^2 - (\frac{1}{r} W' + \frac{1}{r^2} \ddot{W})W''] \quad (28)$$

Let us introduce the following nondimensional quantities:

$$\lambda = 2[3(1-\nu^2)]^{1/4}(R/t)^{1/2}$$

$$x = \frac{\lambda}{a} r$$

$$w = \frac{\lambda^2}{2H} W$$

$$\Psi = \frac{\lambda^4}{4EH^2 t} F$$

$$p = \frac{q}{q_o}$$

where $q_o = \frac{32EH^3 t}{\lambda^2 a^4}$ is the classical buckling pressure of a complete spherical

shell of the same radius of curvature and thickness. Then the nondimensional

forms of Eqs. (27) and (28) are

$$\nabla^4 w = \nabla^2 \Psi + (\frac{1}{x} \Psi' + \frac{1}{x^2} \ddot{\Psi})w'' + (\frac{1}{x} w' + \frac{1}{x^2} \ddot{w})\Psi'' - 2(\frac{1}{x} \dot{\Psi}' - \frac{1}{x^2} \dot{\Psi})(\frac{1}{x} \dot{w}' - \frac{1}{x^2} \dot{w}) + 4p \tag{29}$$

$$\nabla^4 \Psi = -\nabla^2 w + (\frac{1}{x} \dot{w}' - \frac{1}{x^2} \dot{w})^2 - (\frac{1}{x} w' + \frac{1}{x^2} \ddot{w})w'' \tag{30}$$

where $( )' \equiv \frac{\partial}{\partial x}( )$ and $\nabla^2( ) \equiv (\frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} + \frac{1}{x^2} \frac{\partial^2}{\partial \theta^2})( )$.

The edge of the shell is completely clamped; therefore, on the boundary

$r = a$, we have

$$W = 0 \tag{31}$$

$$W' = 0 \tag{32}$$

$$U = 0 \tag{33}$$

$$V = 0 \tag{34}$$

From the last two boundary conditions, we can show by the strain displace-

ment relations that

$$\epsilon_\theta = 0 \tag{35}$$

$$a\epsilon_\theta' - \epsilon_r - \dot{\gamma}_{r\theta} = 0 \tag{36}$$

on the boundary $r = a$.

Expressed in terms of the stress function, these equations become

$$F'' - \frac{\nu}{a} F' - \frac{\nu}{a^2} \ddot{F} = 0 \tag{37}$$

$$a(F'' - \frac{\nu}{r} F' - \frac{\nu}{r^2} \ddot{F})' - \frac{1}{a} F' - \frac{1}{a^2} \ddot{F} + \nu F''' + 2(1+\nu)(\frac{1}{r} \ddot{F})' = 0 \tag{38}$$

on $r = a$.

The nondimensional forms of Eqs. (31), (32), (33) and (34) are

$$w = 0 \tag{39}$$

$$w' = 0 \tag{40}$$

$$Y'' - \frac{\nu}{\lambda} Y' - \frac{\nu}{\lambda^2} \ddot{Y} = 0 \tag{41}$$

$$\lambda(Y'' - \frac{\nu}{x} Y' - \frac{\nu}{x^2} \ddot{Y})' - \frac{1}{\lambda} Y' - \frac{1}{\lambda^2} \ddot{Y} + \nu Y''' + 2(1+\nu)(\frac{1}{x} \ddot{Y})' = 0 \tag{42}$$

on $x = \lambda$.

In addition, at the center of the shell, w and Y must fulfil the requirement of finite stresses.

GOVERNING EQUATIONS FOR BUCKLING OF CLAMPED SHALLOW SPHERICAL SHELLS

As mentioned before, the shell necessarily deforms axisymmetrically only under sufficiently low pressure. The pressure might reach a critical limit (point C in Fig. 2) such that the shell bifurcates from an axisymmetrical deformation path to one of unsymmetrical deformation.

Just after bifurcation, the functions w and Y can be written as

$$w = \omega^*(x) + \omega(x,\theta) \tag{43}$$

$$Y = \psi^*(x) + \psi(x,\theta) \tag{44}$$

where $\omega^*(x)$ and $\psi^*(x)$ are the nondimensional vertical deflection and the

nondimensional stress function just before buckling and hence are axisymmetrical; $\omega(x,\theta)$ and $\psi(x,\theta)$ are due to unsymmetrical buckling and these are considered to be infinitesimally small at the beginning of unsymmetrical buckling. From Eqs. (29) and (30), we have

$$\nabla^4 \omega* = \nabla^2 \psi* + \frac{1}{x} \psi*'\omega*'' + \frac{1}{x} \psi*''\omega*' + 4p \tag{45}$$

$$\nabla^4 \psi* = -\nabla^2 \omega* - \frac{1}{x} \omega*'\omega*'' \tag{46}$$

Put $\theta* = -\omega*'$ and $\phi* = \psi*'$, Eqs. (45) and (46) can be written, after one integration, as

$$(x\theta*') - \frac{\theta*}{x} + x\phi* = -2px^2 + \theta*\phi* \tag{47}$$

$$(x\phi*')' - \frac{\phi*}{x} - x\theta* = -\frac{1}{2}\theta*^2 \tag{48}$$

Also, the boundary conditions can be reduced to

$$\theta*(\lambda) = 0 \tag{49}$$

$$\lambda\phi*'(\lambda) - \nu\phi*(\lambda) = 0 \tag{50}$$

These are the governing equations of the symmetrical problem which has been solved numerically in References 1-4.

Substituting Eqs. (43) and (44) into Eqs. (29) and (30) and using Eqs. (45) and (46), we obtain

$$\nabla^4 \omega = \nabla^2 \psi - (\frac{1}{x}\psi' + \frac{1}{x^2}\ddot{\psi})\theta*' + \frac{1}{x}\omega''\phi* - \frac{1}{x}\psi''\theta*' + (\frac{1}{x}\omega' + \frac{1}{x^2}\ddot{\omega})\phi*' \tag{51}$$

$$\nabla^4 \psi = -\nabla^2 \omega + (\frac{1}{x}\omega' + \frac{1}{x^2}\ddot{\omega})\theta*' + \frac{1}{x}\omega''\theta* \tag{52}$$

where only the linear terms in ω and ψ have been retained. These are two linear differential equations for the functions ω and ψ . Hence our problem has been reduced to an eigenvalue problem where the eigenvalue p is implicit and involved in the solution $\theta*$ and $\phi*$ of the nonlinear problem as defined

by Eqs. (47)-(50).

Let

$$\omega(x,\theta) = \sum_{n=0}^{\infty} \omega_n(x)\cos n\theta \tag{53}$$

$$\psi(x,\theta) = \sum_{n=0}^{\infty} \psi_n(x)\cos n\theta \tag{54}$$

$$L_n(\ ) = (\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{n^2}{x^2})(\ ) \ ; \quad L_n^2(\ ) = L_n L_n(\ )$$

Substituting Eqs. (53) and (54) into (51) and (52), we have, for each n ,

$$L_n^2\omega_n = L_n\psi_n - (\frac{1}{x}\psi_n' - \frac{n^2}{x^2}\psi_n)\theta^{*\prime} + \frac{1}{x}\omega_n''\theta^* - \frac{1}{x}\psi_n''\theta^* + (\frac{1}{x}\omega_n' - \frac{n^2}{x^2}\omega_n)\phi^{*\prime} \tag{55}$$

$$L_n^2\psi_n = -L_n\omega_n + (\frac{1}{x}\omega_n' - \frac{n^2}{x^2}\omega_n)\theta^{*\prime} + \frac{1}{x}\omega_n''\theta^* \tag{56}$$

The boundary conditions can be derived from Eqs. (39)-(42). These are

$$\omega_n(\lambda) = 0 \tag{57}$$

$$\omega_n'(\lambda) = 0 \tag{58}$$

$$\psi_n''(\lambda) - \frac{\nu}{\lambda}\psi_n'(\lambda) + \frac{n^2\nu}{\lambda^2}\psi_n(\lambda) = 0 \tag{59}$$

$$\lambda\psi_n'''(\lambda) - \frac{1}{\lambda}[1 - \nu + (2+\nu)n^2]\psi_n'(\lambda) + \frac{3n^2}{\lambda^2}\psi_n(\lambda) = 0 \tag{60}$$

Equations (55)-(60) are the governing differential equations and the boundary conditions of the problem of unsymmetrical buckling of clamped shallow spherical shells. Existence of a solution for any integer n at a pressure less than the critical pressure of axisymmetrical snapping implies existence of a critical pressure for unsymmetrical buckling.

SOME PRELIMINARY STUDIES

Modified Problems

Before the main problem was treated, it was considered advisable to study some problems of unsymmetrical buckling of shells with modified boundary conditions for which the solutions can be obtained analytically. The purpose of this study is to find whether unsymmetrical buckling rather than axisymmetrical snapping would control in these modified problems. The answers for the critical pressure for unsymmetrical buckling obtained here can be also used to check the accuracy of the numerical procedures for the main problem.

In the first modified problem the edge of the shell is supported by rollers which can slide along a conical wall without changing the slope along the edge of the shell as shown in Figure 5. In the second modified problem the edge condition is the same as that in the first modified problem except that the edge is imagined to be suddenly clamped just before buckling occurs. In both modified problems the shell contracts uniformly before buckling and therefore, $\theta^* = 0$ and $\phi^* = -2px$.

The critical pressures for these modified problems are calculated for $n = 0, 1, 2$ and 3 in Appendix A and are plotted against λ in Figures 5 and 6, where n is the number of waves along the circumferential direction in the buckling mode. In both modified problems it is found that unsymmetrical buckling ($n \neq 0$) dominates for some values of λ . These calculated critical pressures are higher than the classical buckling pressure of a complete spherical shell. In the first modified problem the calculated critical pressures meet the classical buckling pressure periodically at certain values of λ and in the second modified problem the calculated critical pressure approaches the classical

buckling pressure asymptotically when λ approach infinity. In both problems
the results are not realistic because of initial imperfections.

Approximation by Variational Principle

A variational principle is developed in Appendix B for a preliminary study
of the critical pressure of unsymmetrical buckling of clamped shallow spherical
shells. The boundary conditions (57) and (58) of the main problem were specified
in the application of this principle. The differential equations (55) and (56)
are the Euler's equations and the boundary conditions (59) and (60) are the
natural boundary conditions of this variational principle. A Rayleigh-Ritz
method is used in conjunction with the variational principle in Appendix B and
numerical results have been obtained for $\lambda = 6$ as shown in Table 1. These
critical pressures for unsymmetrical buckling are lower than the critical pressure
for axisymmetrical snapping, and this certainly tends to confirm the suspicion
that unsymmetrical buckling does occur for clamped shallow spherical shells.
However, the numerical results obtained by this variational principle are not
necessarily either upper bounds or lower bounds to the exact critical pressure
for unsymmetrical buckling.

NUMERICAL SOLUTION

Difference Equations

The most popular practical method to solve differential equations is to
reduce them to difference equations by confining the range of the independent
variables to a network of mesh points. In this problem we define

$$x = jh \qquad j = 0, 1, \ldots, N = \lambda/h$$

where h is the mesh size.

Let $\omega_j = \omega_n(jh)$; $\psi_j = \psi_n(jh)$; $\theta_j^* = \theta^*(jh)$ and $\phi_j^* = \phi^*(jh)$. Using

$$u_j = \omega_j'' \tag{61}$$

$$v_j = \psi_j'' \tag{62}$$

The differential equations (55) and (56) at $j = 1, 2, \ldots N$ can be written as

$$u_j'' + \frac{2}{x} u_j' - \frac{1+2n^2}{x^2} u_j + \frac{1+2n^2}{x^3} \omega_j' - \frac{4n^2-n^4}{x^4} \omega_j = v_j + \frac{1}{x} \psi_j' - \frac{n^2}{x^2} \psi_j - (\frac{1}{x} \psi_j' - \frac{n^2}{x^2} \psi_j)\theta_j^{*'}$$

$$+ \frac{1}{x} u_j \phi_j^* - \frac{1}{x} v_j \theta_j^* + (\frac{1}{x} \omega_j' - \frac{n^2}{x^2} \omega_j)\phi_j^{*'} \qquad j = 1, 2, \ldots, N \tag{63}$$

$$v_j'' + \frac{2}{x} v_j' - \frac{1+2n^2}{x^2} v_j + \frac{1+2n^2}{x^3} \psi_j' - \frac{4n^2-n^4}{x^4} \psi_j = -u_j - \frac{1}{x} \omega_j' + \frac{n^2}{x^2} \omega_j + (\frac{1}{x} \omega_j' - \frac{n^2}{x^2} \omega_j)\theta_j^{*'}$$

$$+ \frac{1}{x} u_j \theta_j^* \qquad j = 1, 2, \ldots, N \tag{64}$$

The boundary conditions (57)-(60) can be written as

$$\omega_N = 0 \tag{65}$$

$$\omega_N' = 0 \tag{66}$$

$$v_N - \frac{\nu}{\lambda} \psi_N' + \frac{n^2 \nu}{\lambda^2} \psi_N = 0 \tag{67}$$

$$\lambda v_N' - \frac{1}{\lambda} [1 - \nu + (2+\nu)n^2] \psi_N' + \frac{3n^2}{\lambda^2} \psi_N = 0 \tag{68}$$

We shall use the following finite difference approximations for the first and second derivatives:

$$f_j' = \frac{1}{2h} (- f_{j-1} + f_{j+1}) \tag{69}$$

$$f_j'' = \frac{1}{h^2} (f_{j-1} - 2f_j + f_{j+1}) \tag{70}$$

The error of these approximations are $O(h^2)$. Equations (61)-(64) can be

written as

$$A_j y_{j+1} + B_j y_j + C_j y_{j-1} = 0 \qquad\qquad j = 1, 2, \ldots N \qquad (71)$$

where one fictitious station $j = N+1$ has been added off the edge of the shell for convenience. Equations (65)-(68) can be written as

$$G y_{N+1} + K y_N - G y_{N-1} = 0 \qquad (72)$$

where

$$y_j = \begin{bmatrix} \omega_j \\ \Psi_j \\ u_j \\ v_j \end{bmatrix} \qquad (73)$$

$$A_j = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1+2n^2}{2j^3} - \frac{h^2}{2j}\,\theta_j^{*\prime} & -\frac{h^2}{2j}(1-\theta_j^{*\prime}) & h^2(1+\frac{1}{j}) & 0 \\ \frac{h^2}{2j}(1-\theta_j^{*\prime}) & \frac{1+2n^2}{2j^3} & 0 & h^2(1+\frac{1}{j}) \end{bmatrix} \qquad (74)$$

$$B_j = \begin{bmatrix} 2 & 0 & h^2 & 0 \\ 0 & 2 & 0 & h^2 \\ -\frac{4n^2-n^4}{j^4} + \frac{h^2 n^2}{j^2}\,\theta_j^{*\prime} & \frac{h^2 n^2}{j^2}(1-\theta_j^{*\prime}) & -h^2(2+\frac{1+2n^2}{j^2}+\frac{1}{j}\,\theta_j^{*}) & -h^3(h-\frac{1}{j}\,\theta_j^{*}) \\ -\frac{h^2 n^2}{j^2}(1-\theta_j^{*\prime}) & -\frac{4n^2-n^4}{j^4} & h^3(h-\frac{1}{j}\,\theta_j^{*}) & -h^2(2+\frac{1+2n^2}{j^2}) \end{bmatrix} \qquad (75)$$

$$C_j = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{1+2n^2}{2j^3} + \frac{h^2}{2j}\,\theta_j^{*\prime} & \frac{h^2}{2j}(1-\theta_j^{*\prime}) & h^2(1-\frac{1}{j}) & 0 \\ -\frac{h^2}{2j}(1-\theta_j^{*\prime}) & -\frac{1+2n^2}{2j^3} & 0 & h^2(1-\frac{1}{j}) \end{bmatrix} \qquad (76)$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\dfrac{\nu}{2h\lambda} & 0 & 0 \\ 0 & -\dfrac{1}{2h\lambda}[1 - \nu + (2+\nu)n^2] & 0 & \dfrac{\lambda}{2h} \end{bmatrix} \tag{77}$$

and

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \dfrac{n^2\nu}{\lambda^2} & 0 & 1 \\ 0 & \dfrac{3n^2}{\lambda^2} & 0 & 0 \end{bmatrix} \tag{78}$$

From the conditions that all components of stress resultant and moment are bounded at the center of shell, it can be shown that, near $x = 0$ $\omega_n \sim x^n$ and $\Psi_n \sim x^n$. Therefore $C_1 y_0 = 0$, and the first equation of (71) will simply be

$$A_1 y_2 + B_1 y_1 = 0 \tag{79}$$

Determination of Buckling Pressure

Let

$$y_j = Q_j y_{j+1} \tag{80}$$

then

$$y_{j-1} = Q_{j-1} y_j \tag{81}$$

Substituting Eq. (81) into Eq. (71), we get

$$y_j = -(B_j + C_j Q_{j-1})^{-1} A_j y_{j+1}$$

or

$$Q_j = -(B_j + C_j Q_{j-1})^{-1} A_j \tag{82}$$

From Eq. (79)

$$Q_1 = -B_1^{-1}A_1 \tag{83}$$

which also can be obtained from Eq. (82) by setting

$$Q_0 = 0 \tag{84}$$

Equation (82) is a recurrence formula by which all Q's can be evaluated.

Substituting Eq. (80) into Eq. (72), we get

$$Sy_{N+1} = 0 \tag{85}$$

where

$$S = G(I - Q_{N-1}Q_N) + KQ_N \tag{86}$$

and I is a 4x4 unit matrix. For a nontrivial solution $y_{N+1} \neq 0$,

$$|S| = 0 \tag{87}$$

which is the characteristic equation for determination of critical pressures of unsymmetrical buckling.

The case $n = 1$ is a special case where Eqs. (55)-(60) can be satisfied for all p by an exact solution $\omega_1 = 0$ and $\psi_1 = x$. Since the difference equations are exact for linear functions Eqs. (71) and (72) can be satisfied by

$$y_j = \begin{bmatrix} 0 \\ j \\ 0 \\ 0 \end{bmatrix}$$

Equation (85) can be written as

$$[S] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

for n = 1 and so all elements in the second column of matrix S vanish. For n = 1 , Eq. (56) is reduced to

$$\frac{d}{dx} \left[x^2 \psi''' - 3\psi' + \frac{3}{x} \psi + x^2 \omega' - x\omega - (x\omega'-\omega)\theta \right] = 0$$

Integrating from x = 0 to x = λ and using Eqs. (57) and (58), we can obtain Eq. (60). Thus, for n = 1 , Eq. (60) is dependent and can be omitted. We have

$$[\bar{S}] \begin{bmatrix} \omega_{N+1} \\ u_{N+1} \\ v_{N+1} \end{bmatrix} = 0$$

where $\bar{S}$ is the resulting matrix obtained by striking out the second column and the fourth row of the matrix S . For a nontrivial solution,

$$|\bar{S}| = 0 \tag{88}$$

which is the characteristic equation for determination of critical pressures for unsymmetrical buckling for n = 1 .

Numerical Procedure and Results

The numerical procedure of this problem consists of two parts. The first part is to calculate $\theta*$, $\phi*$ and their derivatives for the axisymmetrical deformation before unsymmetrical buckling. We used the iterative method given in Reference (1)[*]. The Poisson's ratio v was taken equal to 1/3 in all calculations and the interval Δx used was 0.25. The first derivatives of $\theta*$ and $\phi*$ were calculated by Eq. (69). The values of Kelvin functions and their first derivatives used were found in Reference (12).

In the second part of the calculation, the elements of matrices A_j , B_j ,

---

[*] The criterion of convergence of the iterative procedure used in this computation is that the final answer for average deflection agrees to four significant figures with the mean of the results of the five previous iterations.

C_j , G and K were calculated first for different values of n and Q_j , S and $|S|$ could be obtained. In general, $|S|$ does not vanish if the trial value of p is not the buckling pressure p_{cr} . The values of $|S|$ were plotted against p . The critical pressure is the lowest root of $|S| = 0$. When n = 1 $\bar{S}$ must be calculated instead of S . This procedure continued for different values of n until axisymmetrical snapping occurred before unsymmetrical buckling appeared.

Two mesh sizes, h = 0.25 and h = 0.125 , were used in the calculation for λ = 6 . Since the mesh size 0.25 was used in the first part of the calculation, in the computation for h = 0.125, the values of θ* , Φ* and their first derivatives at the intermediate stations were obtained by linear interpolation. The difference of buckling pressures calculated by these two mesh sizes was about 0.1 % . Hence we use h = 0.25 in our calculation*.

The validity of this numerical procedure was checked by comparing solutions of the second modified problem in Appendix A (Fig. 6) with solutions obtained by this numerical method. Computations were carried out for the cases λ = 6 and λ = 10 and the maximum error in p_{cr} was 0.4 % .

All numerical calculations were made on the IBM 7090 digital computer and the programming was written in the FORTRAN language. The critical pressures for different values of λ and n are given in Table 2 and also plotted in Figure 8. The lowest value of these critical pressures at a given λ is the governing pressure for unsymmetrical buckling. These governing pressures are shown by a heavy line in Figure 7. Buckling modes were evaluated for these governing

----

* This choice of mesh size provides a reasonable number of stations within the boundary layer of the buckling mode.

pressures and are shown in Figure 8.

According to the results of the calculation, unsymmetrical buckling $(n \neq 0)$ does not occur for $\lambda \leq 5.5$. Unsymmetrical buckling starts to appear when λ is slightly greater than 5.5. As λ keeps increasing the buckling mode shows more and more waves along the circumferential direction and also shows a distinct boundary layer near the edge of the shell along the radial direction when λ is high. An asymptotic analysis has been done in Appendix C where an asymptotic value of the critical pressure for unsymmetrical buckling is found to be 0.864 when λ approaches infinity, and the ratio n/λ is found to approach 0.817.

DISCUSSION

The shell may snap-through under the critical pressure for unsymmetrical buckling if the tangent of the branch on the pressure deflection curve has a negative slope at that critical pressure (as shown by the branch CD in Fig. 2). On the other hand, if that branch has a positive slope (as CE), deformation of the shell changes from the axisymmetrical type to the unsymmetrical type suddenly under the critical pressure but no snapping appears. The determination of the branch of pressure-deflection curve involves the analysis of the post-buckling behavior of the shell.

In Figure 9 the results of previous attempts to calculate critical pressures for buckling of shells are plotted for the purpose of comparison with the present results. Gjelsvik and Bodner[8] and Parmerter and Fung[9] calculated the critical pressures based on an approximate solution for the cases $n = 0$ and $n = 1$. The junctions of curves for $n = 0$ and curves for $n = 1$ in Figure 9 are represented by tick marks. Weinitschke[10] obtained critical pressures for

an extensive range of λ and n by a method which was claimed to be accurate, but the buckling pressures obtained in his work are in serious disagreement with the present results. It is noted that, for $n = 1$ the present result is much closer to the results of References (8) and (9) than Weinitschke's result[*]. In References (7), (8) and (9) the curves representing critical pressures for $n = 0$ are shown to be tangent to those for $n = 1$ at their junctions. However, this tangency does not appear in the curve of present work. The present theoretical buckling pressures are still higher than the experimental results. The effects of initial unsymmetrical geometrical imperfections and variation of shell thickness are presumed to be the source of this discrepancy, but the analysis of such problem is very complicated.

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[*] More recently, in a private communication, G. A. Thurston stated that he found a lower bound to the critical pressure $p_{cr} = 0.753$ for $n = 4$ and $\lambda = 8$, which is close to the present result $p_{cr} = 0.766$ (Table 2).

APPENDIX A

Analysis of Shells with Modified Boundary Conditions

As mentioned before, in both modified problems

$$\theta^* = 0 \tag{A1}$$

$$\Phi^* = -2px \tag{A2}$$

Substituting them into Eqs. (55) and (56) one obtains

$$L_n^2 \omega_n = L_n \psi_n - 2pL_n \omega_n \tag{A3}$$

$$L_n^2 \psi_n = -L_n \omega_n \tag{A4}$$

The general solutions are expressed in terms of Bessel's functions

$$\omega_n = A_n J_n(\mu x) + B_n J_n(\tfrac{x}{\mu}) + C_n x^n + E_n Y_n(\mu x) + F_n Y_n(\tfrac{x}{\mu}) + G_n x^{-n} \tag{A5}$$

$$\psi_n = \frac{A_n}{\mu^2} J_n(\mu x) + B_n \mu^2 J_n(\tfrac{x}{\mu}) + D_n x^n + \frac{E_n}{\mu^2} Y_n(\mu x) + F_n \mu^2 Y_n(\tfrac{x}{\mu}) + H_n x^{-n} \tag{A6}$$

where

$$p = \frac{1}{2} (\mu^2 + \frac{1}{\mu^2}) \tag{A7}$$

All components of stress resultant and moment are bounded at the center of the shell, therefore,

$$\omega_n = A_n J_n(\mu x) + B_n J_n(\tfrac{x}{\mu}) + C_n x^n \tag{A8}$$

$$\psi_n = \frac{A_n}{\mu^2} J_n(\mu x) + B_n \mu^2 J_n(\tfrac{x}{\mu}) + D_n x^n \tag{A9}$$

The boundary conditions of the first modified problem are conveniently obtained from the principle of virtual work

$$\int_0^{2\pi} \int_0^a (M_r \delta \kappa_r + M_\theta \delta \kappa_\theta + M_{r\theta} \delta \kappa_{r\theta} + N_r \delta \epsilon_r + N_\theta \delta \epsilon_\theta + N_{r\theta} \delta \gamma_{r\theta} - q \delta W) r \, dr d\theta = 0 \tag{A10}$$

Using the strain displacement relations we get

$$- \int_0^{2\pi} \int_0^a \left\{ \left[(rM_r)'' + \left(\frac{\ddot{M}_\theta}{r}\right) - M'_\theta + \frac{2}{r} (r\dot{M}_{r\theta})' - (rN_r z')' - \left(\frac{\overline{N_\theta \dot{z}}}{r}\right) - (N_{r\theta} \dot{z})' - (\overline{N_{r\theta} z'})' + qr \right] \delta W \right.$$

$$\left. + \left[(rN_r)' + \dot{N}_{r\theta} - N_\theta \right] \delta U + \left[(rN_{r\theta})' + \dot{N}_\theta + N_{r\theta} \right] \delta V \right\} r \, dr d\theta$$

$$+ \int_0^{2\pi} \left\{ \left[(rM_r)' + 2\dot{M}_{r\theta} - M_\theta \right] \delta \zeta + \left[\frac{2H}{a} \left((rM_r)' + 2\dot{M}_{r\theta} - M_\theta + 2HN_r \right) + aN_r \right] \delta \xi - aM_r \delta W' \right.$$

$$\left. + aN_{r\theta} \delta V \right\}_{r=a} d\theta = 0 \tag{A11}$$

where ζ and ξ are the displacements along the wall and perpendicular to the wall respectively, therefore,

$$\zeta = \frac{2H}{a} U - W \tag{A12}$$

$$\xi = U + \frac{2H}{a} W \tag{A13}$$

By substitution, we get the boundary conditions

$$(rM_r)' + 2\dot{M}_{r\theta} - M_\theta = 0 \tag{A14}$$

$$W' = 0 \tag{A15}$$

$$\xi = 0 \tag{A16}$$

$$N_{r\theta} = 0 \tag{A17}$$

In terms of the nondimensional quantities, these are

$$\omega'_n = 0 \tag{A18}$$

$$\lambda (L_n \omega_n)' + (1-\nu) \frac{n^2}{\lambda^2} \omega_n = 0 \tag{A19}$$

$$\lambda \psi'_n - \psi_n = 0 \tag{A20}$$

$$L_n \psi_n - \lambda (L_n \psi_n)' + (1+\nu)(n^2-1)\psi_n + \omega_n = 0 \tag{A21}$$

Substituting Eqs. (A8) and (A9) into Eqs. (A18)-(A21) and eliminating the coefficient of D_n , we have the following characteristic equation from the resulting equations

$$
\begin{vmatrix}
\mu\lambda J_n'(\mu\lambda) & \dfrac{\lambda}{\mu} J_n'(\tfrac{\lambda}{\mu}) & n \\[2mm]
-\mu^3\lambda^3 J_n'(\mu\lambda) + (1-\nu)n^2 J_n(\mu\lambda) & -\dfrac{\lambda^3}{\mu^3} J_n'(\tfrac{\lambda}{\mu}) + (1-\nu)n^2 J_n(\tfrac{\lambda}{\mu}) & (1-\nu)n^2 \\[2mm]
\dfrac{1}{\mu\lambda} J_n'(\mu\lambda) - \dfrac{n}{\mu^2\lambda^2} J_n(\mu\lambda) & \dfrac{\mu}{\lambda} J_n'(\tfrac{\lambda}{\mu}) - n \dfrac{\mu^2}{\lambda^2} J_n(\tfrac{\lambda}{\mu}) & \dfrac{n-1}{n+1}\dfrac{1}{(1+\nu)\lambda^2}
\end{vmatrix} = 0 \quad (A22)
$$

The case $n = 1$ is also a special case in the modified problems. However, Eq. (A22) still holds for $n = 1$.

The buckling pressures are plotted against λ for different values of n in Figure 5. The case $n = 0$ represents an axisymmetrical buckling.

The boundary conditions of the second modified problem are the same as those of the main problem. Substituting Eqs. (A8) and (A9) into Eqs. (57)-(60) and eliminating the coefficient of D_n , we get the following characteristic equation

$$
\begin{vmatrix}
J_n(\mu\lambda) & J_n(\tfrac{\lambda}{\mu}) & 1 \\[2mm]
\mu\lambda J_n'(\mu\lambda) & \dfrac{\lambda}{\mu} J_n'(\tfrac{\lambda}{\mu}) & n \\[2mm]
\dfrac{n}{\mu\lambda} J_n'(\mu\lambda) - \dfrac{n(1+n)}{2\mu^2\lambda^2} J_n(\mu\lambda) & \dfrac{n\mu}{\lambda} J_n'(\tfrac{\lambda}{\mu}) - \dfrac{n(1+n)\mu^2}{2\lambda^2} J_n(\tfrac{\lambda}{\mu}) & -\dfrac{1}{1+\nu}
\end{vmatrix} = 0 \quad (A23)
$$

This equation holds for the special case $n = 1$ but is not applicable for $n = 0$ where we can show by using Eqs. (55) and (56) that Eqs. (57)-(60) are linearly dependent. Omitting the last boundary condition equation (60), we can get the characteristic equation for $n = 0$, which is

$$
\mu\lambda J_0(\tfrac{\lambda}{\mu}) J_0'(\mu\lambda) - \dfrac{\lambda}{\mu} J_0(\mu\lambda) J_0'(\tfrac{\lambda}{\mu}) + (1+\nu)(\mu^2 - \dfrac{1}{\mu^2}) J_0'(\mu\lambda) J_0'(\tfrac{\lambda}{\mu}) = 0 \quad (A24)
$$

The $P_{cr} - \lambda$ curves for the second modified problem are shown in Figure 6.

APPENDIX B

Variational Principle for Buckling of Clamped Shallow Spherical Shells

A variational principle for the problem of clamped shallow spherical shells was first considered by Weinitschke[13] for the problem of axisymmetrical deformation of shells. From a modified Reissner variational principle[14], we can show that Eqs. (45), (46), (49) and (50) are equivalent to the following variational principle:

$$\delta \int_0^\lambda \left\{ \frac{1}{x} \psi^{*\prime}[x\omega^{*\prime}+\frac{1}{2}(\omega^{*\prime})^2]+ \frac{1}{2}(\nabla^2\omega^*)^2- \frac{1}{2}(\nabla^2\psi^*)^2-4p\omega^* \right\} x\,dx+ \frac{1}{2}(1+\nu)\delta[\psi^{*\prime}(\lambda)]^2 = 0$$

(B1)

where $\omega^*(\lambda) = \omega^{*\prime}(\lambda) = 0$ is specified. Furthermore, Eqs. (55)-(60) are equivalent to the following variational principle:

$$\delta \int_0^\lambda \left\{ \frac{1}{2x} \psi^{*\prime}(\omega_n^\prime)^2+ \frac{1}{2} \psi^{*\prime\prime} \frac{n^2}{x^2}(\omega_n)^2+(\frac{1}{x} \psi_n^\prime- \frac{n^2}{x^2} \psi_n)(x\omega_n^\prime+\omega^{*\prime}\omega_n^\prime)-n^2(\frac{1}{x} \psi_n)^\prime(\omega_n+ \frac{\omega_n\omega^{*\prime}}{x}) \right.$$

$$\left. + \frac{1}{2}(L_n\omega_n)^2- \frac{1}{2}(L_n\psi_n)^2 \right\} x\,dx+ \frac{1}{2}(1+\nu)\delta[(\psi_n^\prime)^2-n^2(\frac{\psi_n^2}{x})^\prime]_{x=\lambda} = 0 \qquad (B2)$$

where $\omega_n(\lambda) = \omega_n^\prime(\lambda) = 0$ is specified and variations are taken with respect to ω_n and ψ_n.

In order to apply these variational principles we try

$$\omega^* = (x^2-\lambda^2)(Ax^2+B\lambda^2)/\lambda^4 \qquad (B3)$$

$$\psi^* = Cx^4 + D\lambda^2x^2 \qquad (B4)$$

$$\omega_n = Ex^n(x^2-\lambda^2)^2 \qquad (B5)$$

$$\psi_n = x^n(Fx^4+G\lambda^2x^2+H\lambda^4) \qquad (B6)$$

Substituting these expressions into Eq. (B1) and taking variations with

respect to A , B , C and D , we obtain four nonlinear equations for the
unknowns A , B , C and D ; their values can be calculated numerically for
any assigned p . Substituting Eqs. (B3)-(B6) into Eq. (B2) and taking
variations with respect to E , F , G and H , we obtain four homogeneous
linear equations and hence the characteristic equation from which the buckling
pressure can be determined for any n . The numerical calculation has been done
for $\lambda = 6$ and the buckling pressures are shown in Table 1.

APPENDIX C

Asymptotic Solution for Buckling Pressure for

Shallow Spherical Shells with Large λ

It is interesting to consider the asymptotic behavior of clamped shallow spherical shells when λ approaches infinity. A boundary layer is found near the edge of the shell when the shell deforms axisymmetrically before buckling. In the region outside the boundary layer the shell deforms essentially by a rigid body downward displacement, hence all components of stress resultant and moment, except N_r , vanish.

Let

$$\bar{\Phi} = \Phi^* + 2px \tag{C1}$$

Equations (47)-(50) can be written as

$$\theta^{*\prime\prime} + \frac{\theta^{*\prime}}{x} - \frac{\theta^*}{x^2} + \bar{\Phi} = -2p\theta^* + \frac{\bar{\Phi}\theta^*}{x} \tag{C2}$$

$$\bar{\Phi}^{\prime\prime} + \frac{\bar{\Phi}^{\prime}}{x} - \frac{\bar{\Phi}}{x^2} - \theta^* = -\frac{1}{2x}\theta^{*2} \tag{C3}$$

$$\theta^*(\lambda) = 0 \tag{C4}$$

$$\bar{\Phi}^{\prime}(\lambda) - \frac{\nu}{\lambda}\bar{\Phi}(\lambda) = 2(1-\nu)p \tag{C5}$$

In the region of boundary layer where x approaches infinity, the following equations can be used:

$$\theta^{*\prime\prime} + \bar{\Phi} = -2p\theta^* \tag{C6}$$

$$\bar{\Phi}^{\prime\prime} - \theta^* = 0 \tag{C7}$$

$$\theta^*(\lambda) = 0 \tag{C8}$$

$$\bar{\Phi}^{\prime}(\lambda) = 2(1-\nu)p \tag{C9}$$

When $p < 1$ this problem can be solved analytically. The final results are

$$\theta* = \frac{2\sqrt{2p}\ (1-\nu)}{\sqrt{1+p}}\ e^{-\sqrt{\frac{1-p}{2}}\ (\lambda-x)}\ \sin\sqrt{\frac{1+p}{2}}\ (\lambda-x) \tag{C10}$$

$$\bar{\Phi} = \frac{2\sqrt{2p}\ (1-\nu)}{\sqrt{1+p}}\ e^{-\sqrt{\frac{1-p}{2}}\ (\lambda-x)}\ [\sqrt{1-p^2}\ \cos\sqrt{\frac{1+p}{2}}\ (\lambda-x) - p\ \sin\sqrt{\frac{1+p}{2}}\ (\lambda-x)]$$

or

$$\Phi* \approx -2px \tag{C11}$$

From Figure 8 it can be seen that a boundary layer also appears in the buckling mode of unsymmetrical buckling. Let δ_b be an effective boundary layer thickness and

$$\bar{x} = x + \delta_b - \lambda \tag{C12}$$

$$(\ )' = \frac{d}{d\bar{x}}\ (\ )$$

then, in the region of boundary layer, Eqs. (55)-(60) become

$$\omega'''' - 2\sigma\ \omega'' + \sigma^2\omega - \psi'' + \sigma(1-\theta*')\psi + 2p\omega'' - 2p\sigma\ \omega = 0 \tag{C13}$$

$$\psi'''' - 2\sigma\ \psi'' + \sigma^2\psi + \omega'' - \sigma(1-\theta*')\omega = 0 \tag{C14}$$

$$\omega(\delta_b) = 0 \tag{C15}$$

$$\omega'(\delta_b) = 0 \tag{C16}$$

$$\psi''(\delta_b) + \nu\ \sigma\ \psi(\delta_b) = 0 \tag{C17}$$

$$\psi'''(\delta_b) - (2+\nu)\sigma\ \psi'(\delta_b) = 0 \tag{C18}$$

where

$$\sigma = \lim_{\lambda\to\infty} (\frac{n}{\lambda})^2 \tag{C19}$$

and

$$\theta^{*\prime} = 2p(1-\nu)e^{-\sqrt{\frac{1-p}{2}}\,(\delta-\bar{x})} \left[\sqrt{\frac{1-p}{1+p}} \sin\sqrt{\frac{1+p}{2}}\,(\delta-\bar{x}) - \cos\sqrt{\frac{1+p}{2}}\,(\delta-\bar{x}) \right] \qquad (C20)$$

Also

$$\omega(0) = \omega''(0) = \psi(0) = \psi''(0) = 0 \qquad (C21)$$

By the same method as given in the main problem the asymptotic value of the critical pressures can be evaluated numerically. In the calculation, δ_b was chosen equal to 40. The critical pressures p_{cr} are plotted against σ in Figure 10. The minimum of this curve determines the required asymptotic value of the critical pressure for λ equal to infinity which is found to be $p_{cr} = 0.864$ when the ratio n/λ approaches 0.817. The buckling mode is also calculated and is shown in Figure 11 from which the actual boundary layer thickness is found to be 12 approximately.

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n	P_{cr} calculated by variational principle
1	> 0.85
2	0.770
3	0.751
4	0.850

Buckling Pressures Calculated by Variational Principle for $\lambda = 6$

Table 1

n	λ																
	3.5	4	5	5.5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0.614	0.578	0.629	0.762 ⏐ 0.763	0.995	1.068	1.130	0.931 ⏐ 0.94	0.822 ⏐ 0.83	0.83 ⏐ 0.84	0.96 ⏐ 0.97	0.96 ⏐ 0.97	0.98 ⏐ 0.99	0.98 ⏐ 0.99	0.92 ⏐ 0.93	0.90 ⏐ 0.91	0.94 ⏐ 0.95
1					0.919	1.023											
2					0.775	0.796	0.893										
3					0.827	0.760	0.774	0.846									
4					0.931	0.812	0.766	0.777	0.814		0.899						
5						0.902	0.813	0.777	0.776	0.800	0.841	0.872	0.900	0.929			
6						1.000	0.887	0.816	0.780	0.776	0.798	0.826	0.850	0.874	0.905		
7							0.973	0.877	0.812	0.782	0.780	0.794	0.814	0.835	0.859	0.879	0.906
8							1.057			0.810	0.790	0.781	0.790	0.807	0.828	0.847	0.861
9											0.815	0.792	0.782	0.790	0.803	0.818	0.838
10											0.853	0.816	0.801	0.785	0.792	0.800	0.818
11											0.896	0.849	0.815	0.797	0.790	0.792	0.802
12											0.942	0.892	0.843	0.814	0.800	0.792	0.793
13												0.928	0.876	0.843	0.818	0.800	0.794
14													0.918	0.873	0.842	0.814	0.803
15													0.951	0.911	0.872	0.837	0.818
16														0.945	0.901	0.864	0.835
17														0.976		0.892	0.860
18																	0.888
19																	0.911

CALCULATED BUCKLING PRESSURES OF CLAMPED SHALLOW SPHERICAL SHELLS

TABLE 2
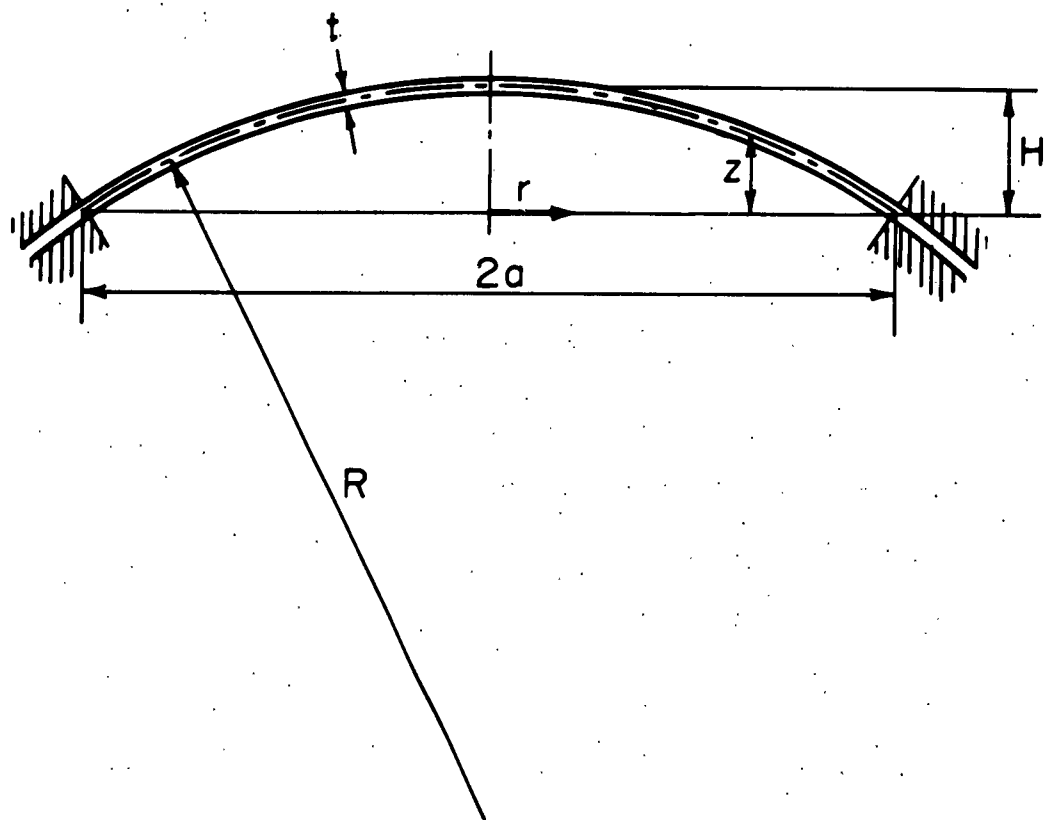
FIG. I GEOMETRY OF CLAMPED SHALLOW SPHERICAL SHELLS
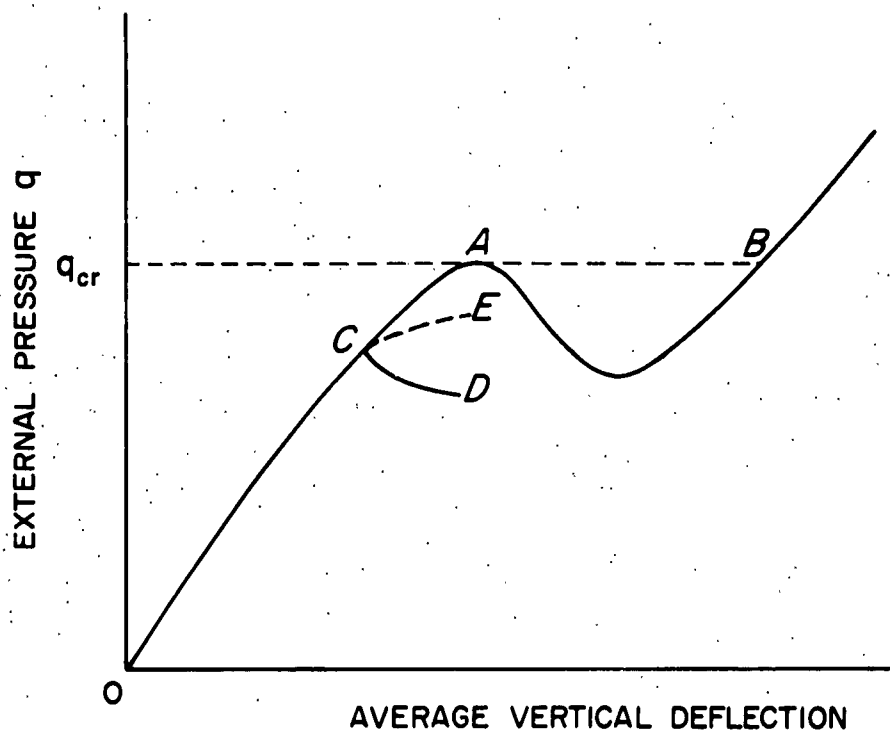
FIG. 2   PRESSURE-DEFLECTION CURVE OF CLAMPED
SHALLOW SPHERICAL SHELLS

q_{cr} = CRITICAL PRESSURE OF AXISYMMETRICAL SNAPPING

q_o = CLASSICAL BUCKLING PRESSURE OF COMPLETE SPHERICAL SHELL

$\rho_{cr} = \dfrac{q_{cr}}{q_o}$

$\lambda = 2[3(1-\nu^2)]^{1/4}\left(\dfrac{H}{t}\right)^{1/2}$

EXPERIMENTS

△ KAPLAN & FUNG

○ HOMEWOOD et al.
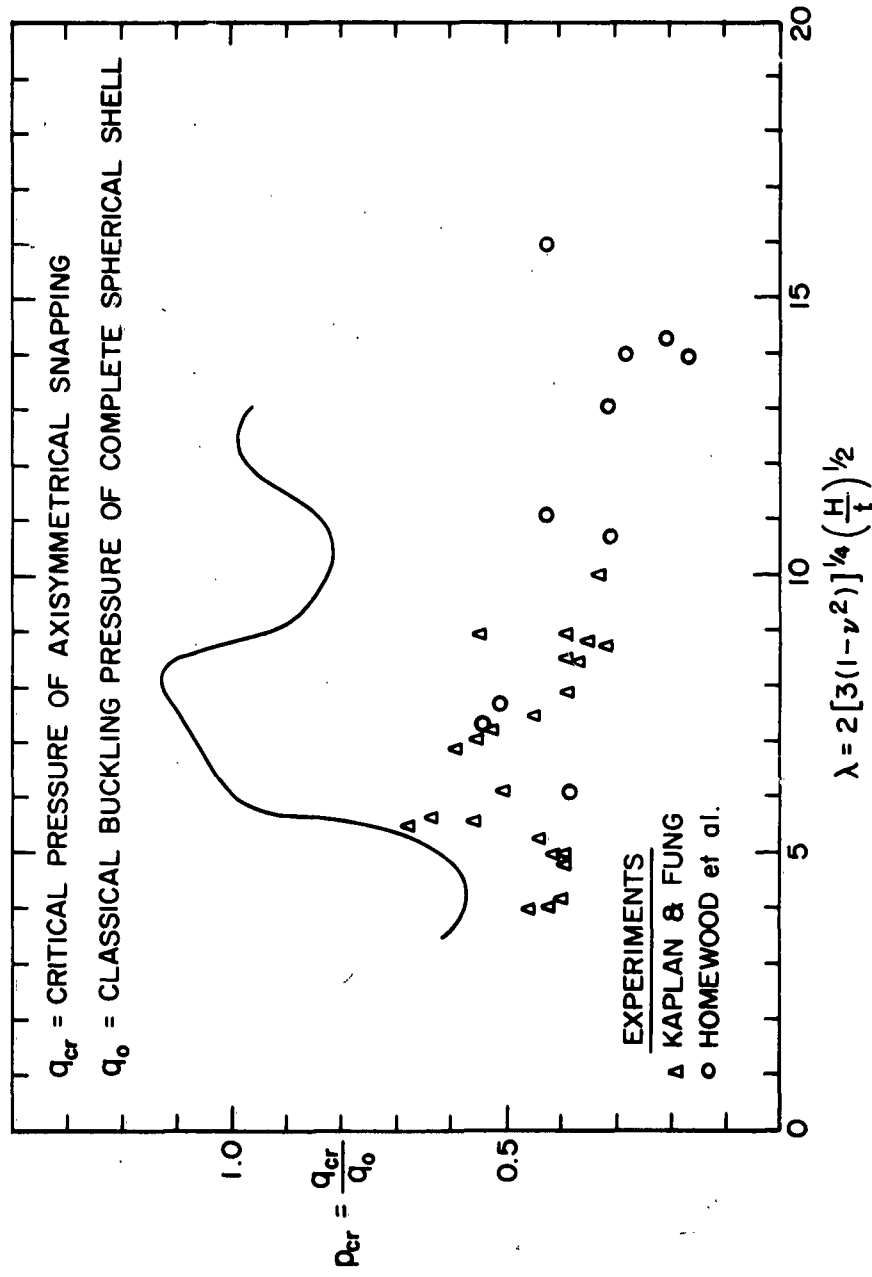
FIG. 3 CALCULATED CRITICAL PRESSURES FOR AXISYMMETRICAL SNAPPING AND EXPERIMENTAL RESULTS
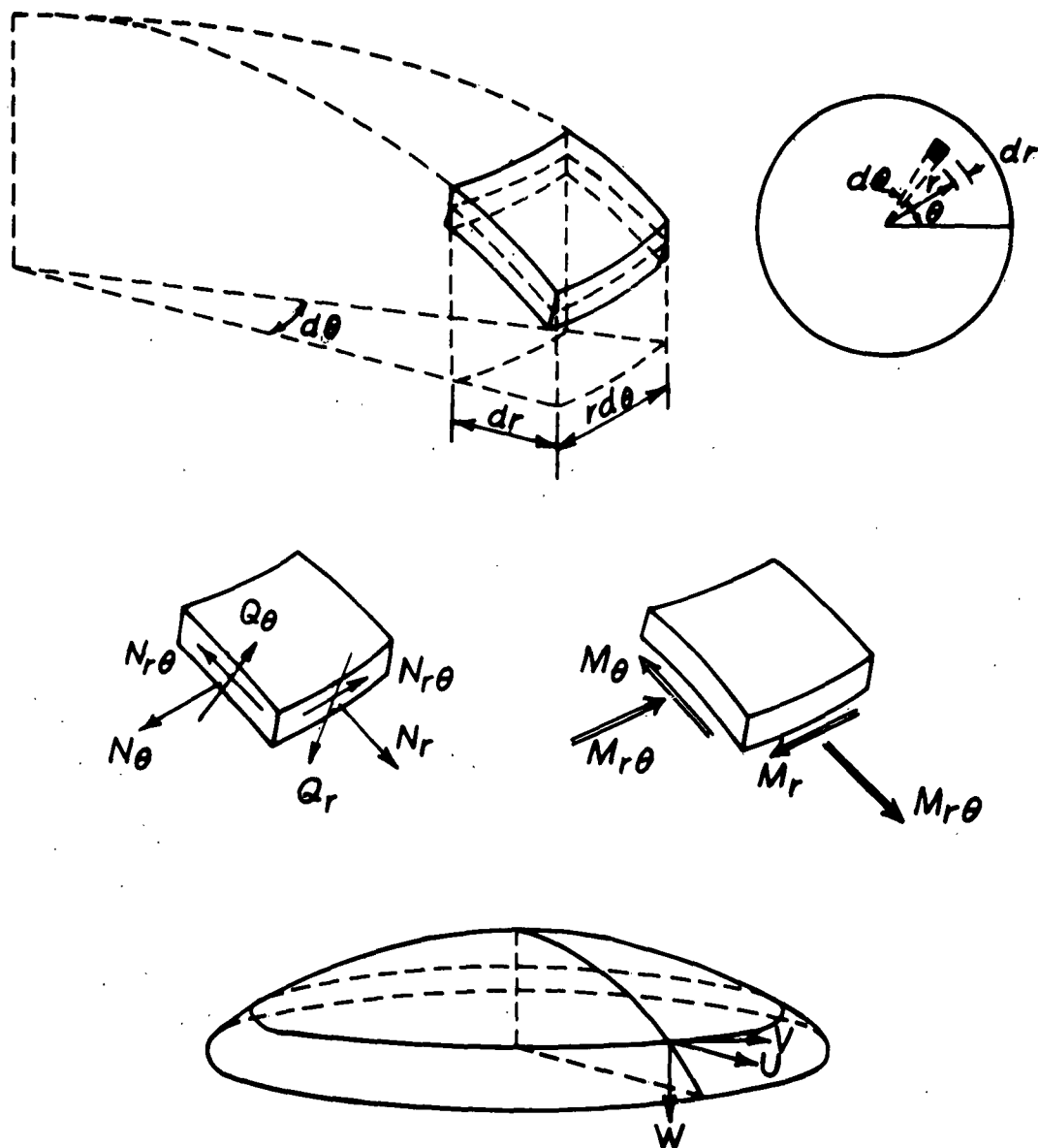
FIG. 4   STRESS RESULTANTS, MOMENTS AND
DISPLACEMENTS IN SHELLS
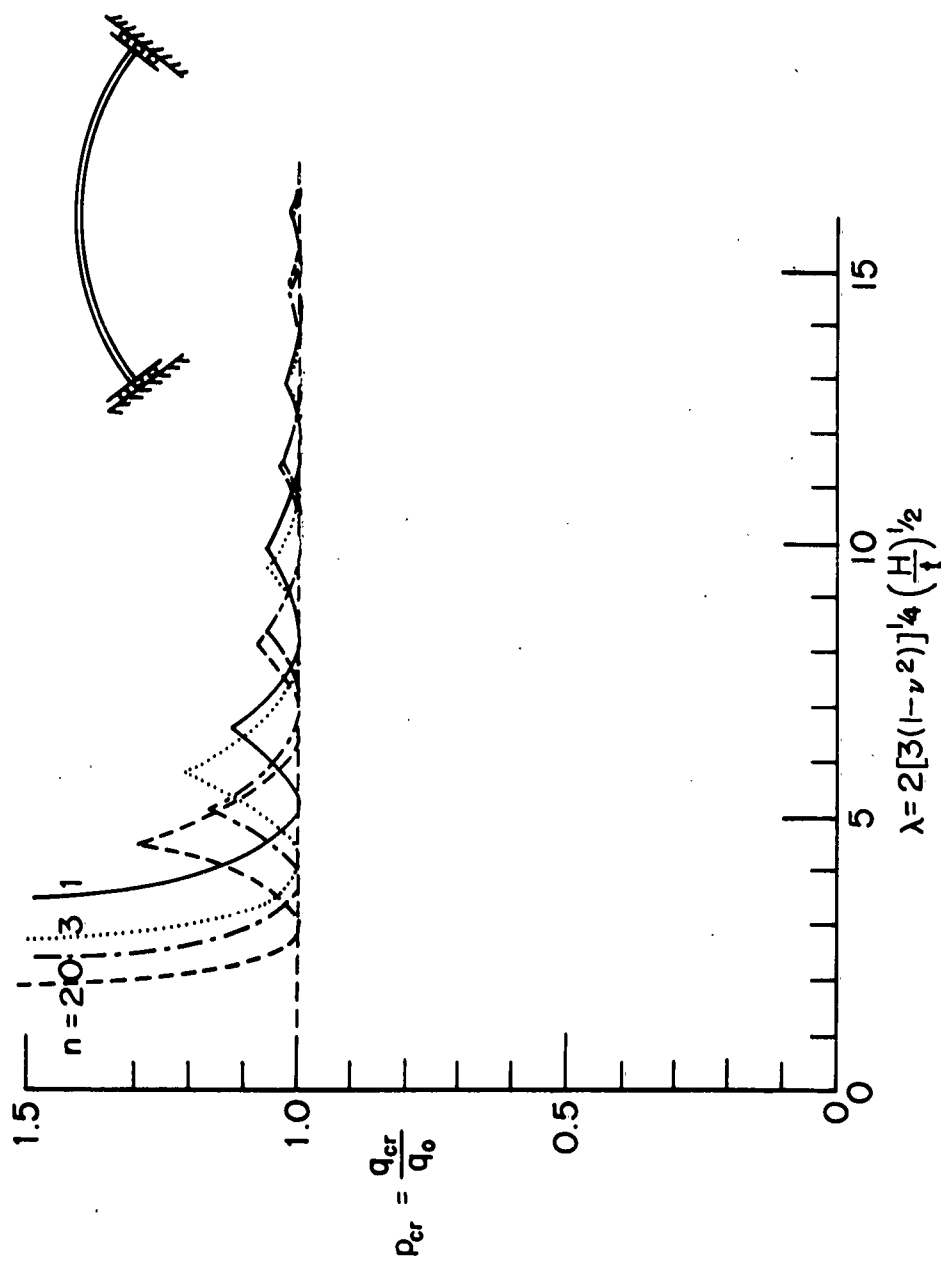
FIG. 5   BUCKLING PRESSURES IN THE FIRST MODIFIED PROBLEM
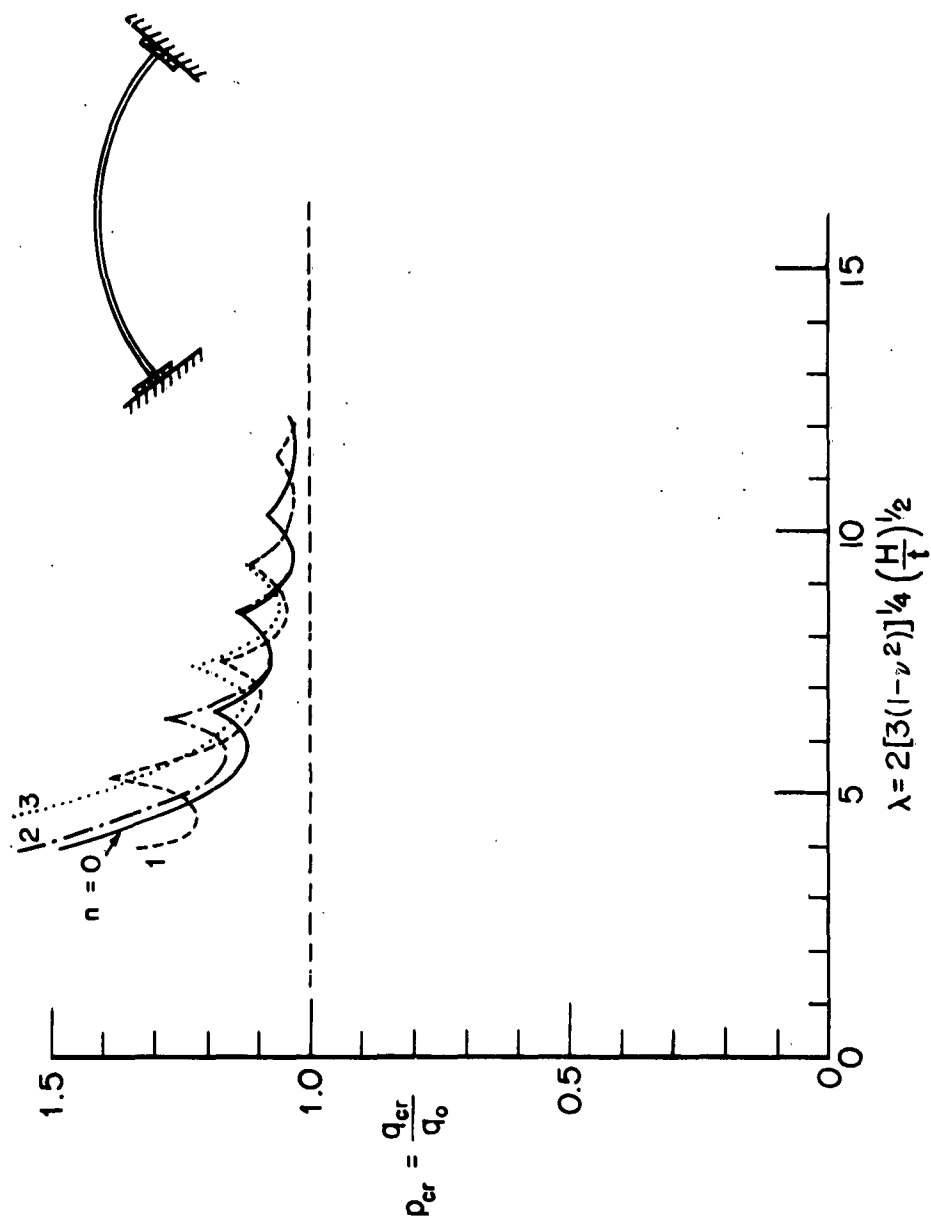
FIG. 6   BUCKLING PRESSURES IN THE SECOND MODIFIED PROBLEM
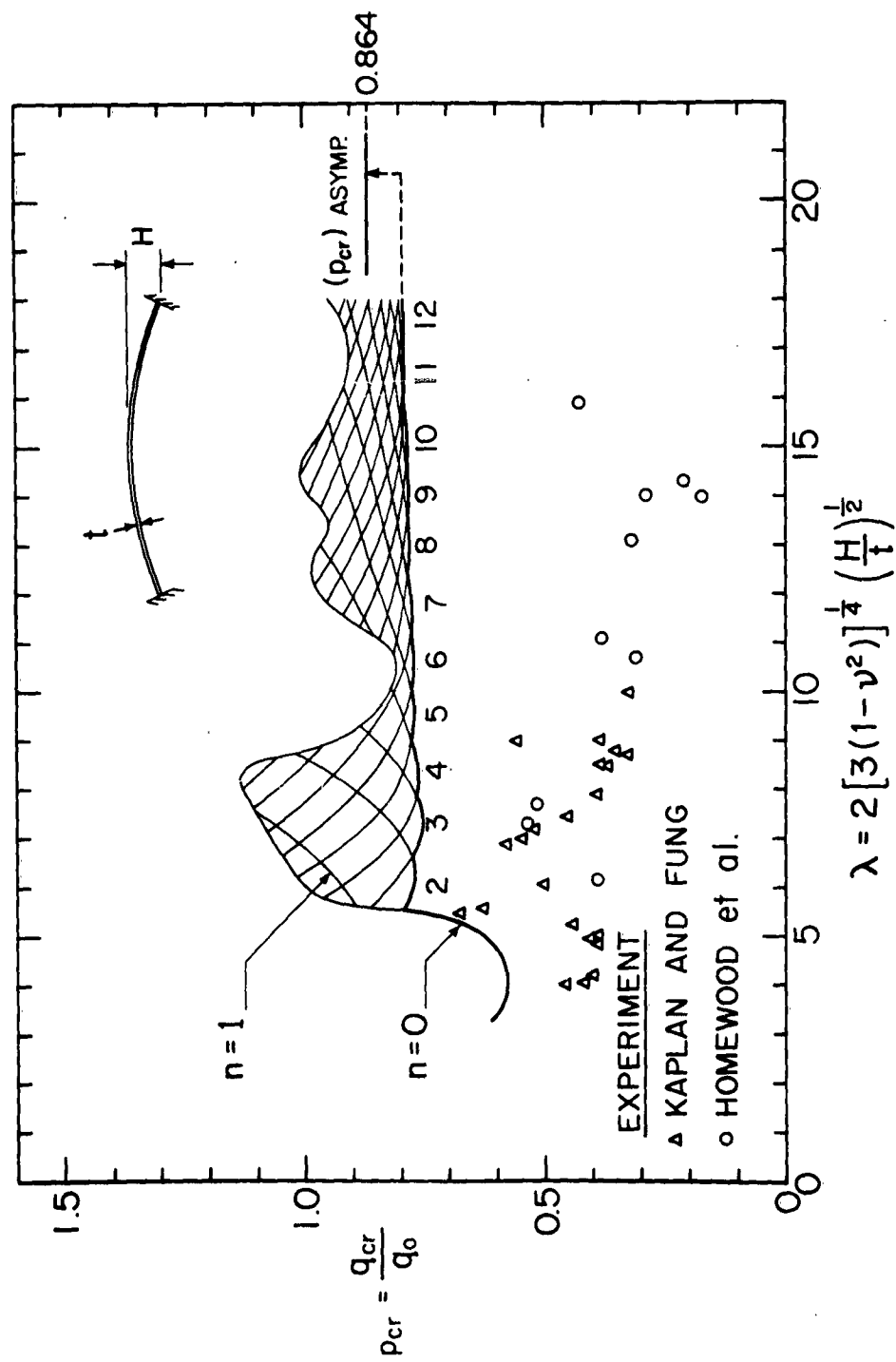
FIG. 7 CALCULATED BUCKLING PRESSURES OF CLAMPED SHALLOW
SPHERICAL SHELLS AND EXPERIMENTAL RESULTS

$$\lambda = 2\left[3\left(1-\nu^2\right)\right]^{\frac{1}{4}}\left(\frac{H}{t}\right)^{\frac{1}{2}}$$

$$p_{cr} = \frac{q_{cr}}{q_o}$$

EXPERIMENT

△ KAPLAN AND FUNG

○ HOMEWOOD et al.

(p_{cr}) ASYMP. 0.864

n = 1

n = 0
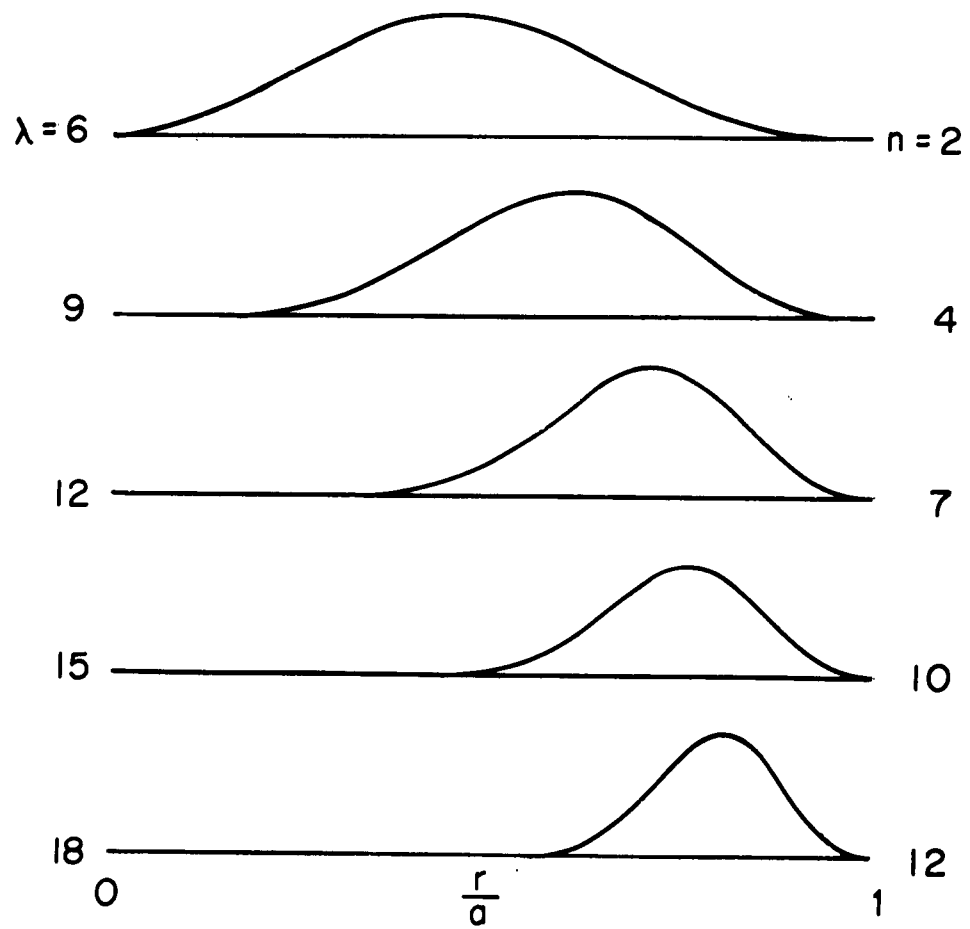
FIG. 8 BUCKLING MODES OF CLAMPED SHALLOW
SPHERICAL SHELLS
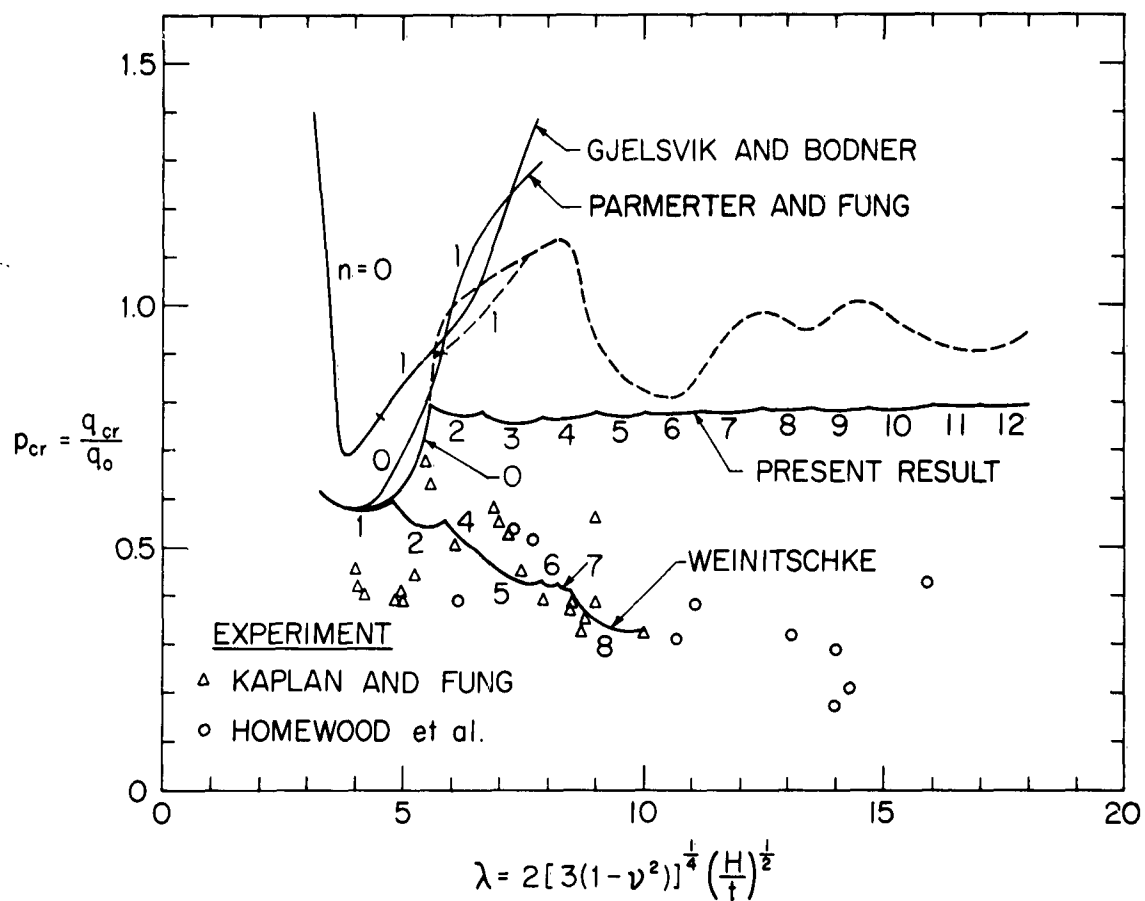
FIG. 9 COMPARISON OF CALCULATED PRESSURES OF UNSYMMETRICAL
BUCKLING WITH PREVIOUS RESULTS

FIG. 10 $p_{cr} - \sigma$ CURVE

← TO x = 0

x = λ
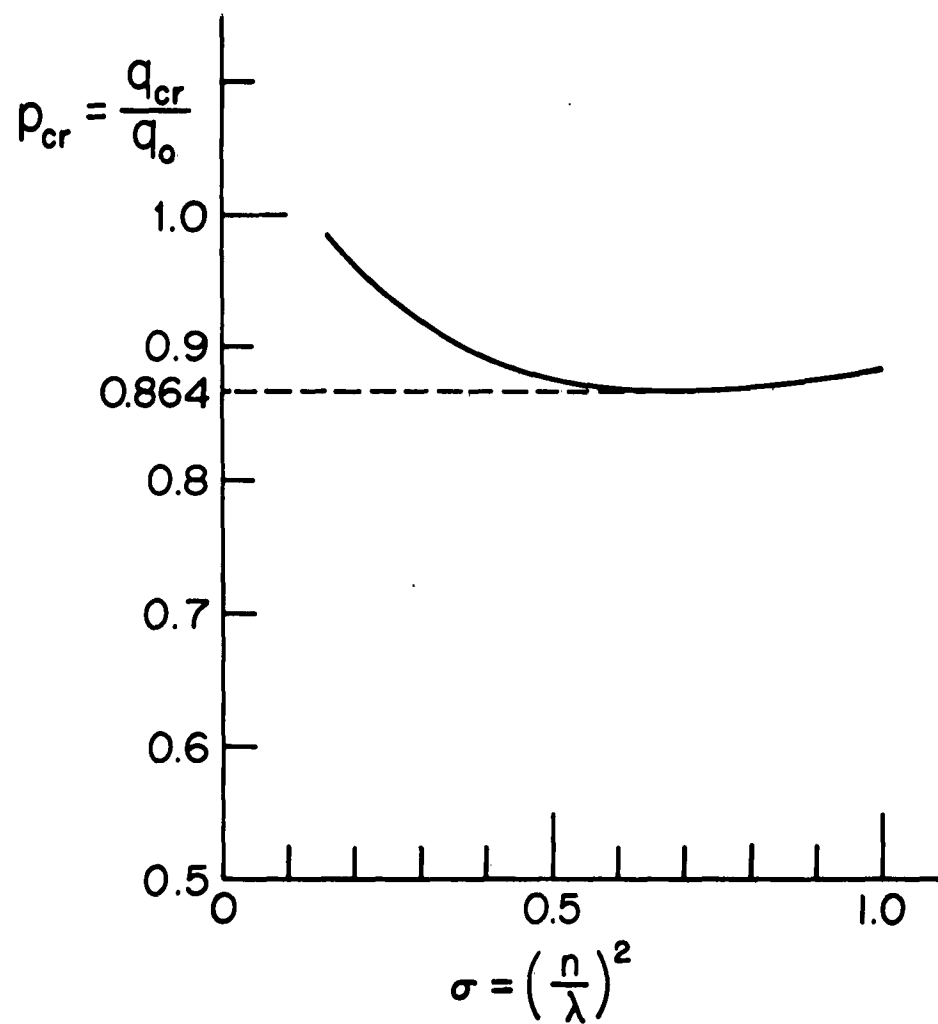
FIG. 11 BUCKLING MODE OF CLAMPED SHALLOW SPHERICAL SHELLS
WHEN λ → ∞